



Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

An Isoperimetric Problem with Variable End-Points.

BY ARCHIBALD SHEPARD MERRILL.

Introduction.

The object of this paper is to give a complete discussion of the necessary and sufficient conditions for a maximum (minimum) for a type of problems in the Calculus of Variations which are closely related to the usual isoperimetric problems, and in which both end-points are allowed to vary along a given fixed curve. We suppose that we have given the fixed curve L and a certain arc E_{12}

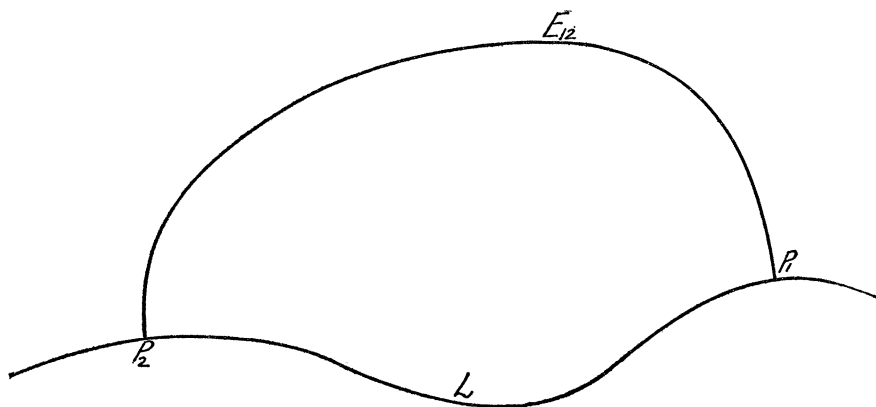


FIG. 1.

joining two points P_1 and P_2 of L . The problem is to determine the properties which the curve E_{12} must have in order that the integral of a given function $F(x, y, x', y')$ along L from P_2 to P_1 , then along E_{12} from P_1 to P_2 shall be a maximum (minimum), while the integral of a second function $G(x, y, x', y')$ along E_{12} has a prescribed value. Thus the function J to be maximized (minimized) is a sum of two integrals:

$$J = \int_{L_{21}} F(x, y, x', y') dx + \int_{E_{12}} F(x, y, x', y') dt,$$

while the integral

$$K = \int_{E_{12}} G(x, y, x', y') dt$$

is to remain fixed in value.

A familiar application of this type of problem is the well-known Problem of Dido. In this application the area included by the arc P_1P_2 of E_{12} and the arc P_2P_1 of L is to be a maximum, while the arc P_1P_2 of E_{12} is to have a pre-assigned length.

For the problem at hand certain conditions of the corresponding isoperimetric problem with fixed end-points must hold, and these are already well known, viz., the Euler, Weierstrass, Legendre and Jacobi necessary conditions.* The transversality condition is also readily obtainable, and has been deduced for some special cases of the problem here discussed.† In the present paper a new necessary condition, corresponding to the Jacobi condition in other problems in the Calculus of Variations, will be deduced and discussed both for the case of one end-point variable and for the case of both end-points variable. In obtaining this we make use of the derivatives of the "extremal integral" for any isoperimetric problem, and Section 2 is given over entirely to the computation of these derivatives. Conditions which are sufficient for a maximum of J when K is fixed are readily obtained with the help of a theorem proved by Hahn. A geometric interpretation of the new condition is given in Section 5. Finally in Section 6, as an application of the general theory developed, a discussion of the above-mentioned Problem of Dido is given.

§ 1. *Conditions Deducible from Known Results.*

Consider a fixed curve

$$x=\tilde{x}(\kappa), \quad y=\tilde{y}(\kappa) \tag{L}$$

not intersecting itself, and two points $P_1(\kappa=\kappa_1)$ and $P_2(\kappa=\kappa_2)$ on L with $\kappa_2 < \kappa_1$. Let E_{12} be an arc

$$x=\phi(t), \quad y=\psi(t), \quad t_1 \leq t \leq t_2, \tag{E}$$

cutting L at $P_1(t=t_1)$ and $P_2(t=t_2)$. The function to be maximized or minimized is then of the form

$$J = \int_{\kappa_2}^{\kappa_1} F(\tilde{x}, \tilde{y}, x', y') d\kappa + \int_{t_1}^{t_2} F(\phi, \psi, \phi', \psi') dt,$$

while the integral

$$K = \int_{t_1}^{t_2} G(\phi, \psi, \phi', \psi') dt$$

is to remain constant in value.

For simplicity the following discussion will be restricted to the determination of a maximum for J . Consider the totality of arcs whose end-points

* See Bolza, "Vorlesungen über Variationsrechnung, Chapter X.

† See Bolza, *loc. cit.*, p. 520.

lie upon L . In this class there is a sub-class \mathfrak{M} of arcs which give the integral K a fixed value k . The problem is then to find necessary and sufficient conditions that a particular arc E_{12} of \mathfrak{M} , intersecting arc L at $P_1(\kappa=\kappa_1)$ and $P_2(\kappa=\kappa_2)$, shall give to J a larger value than any other arc of \mathfrak{M} in a certain neighborhood of arc E_{12} .

It is presupposed that the arc L is regular* in a neighborhood of the values $\kappa_2 \leq \kappa \leq \kappa_1$. The class \mathfrak{M} is further restricted to contain only regular arcs, and in particular the arc E_{12} whose maximizing properties are to be investigated is assumed to be of class C''' .† It is also assumed that E is not an extremal for the integral K .

The function G is of class C''' for all values (x, y, x', y') for which $(x', y') \neq (0, 0)$ and (x, y) is in a neighborhood of E_{12} , while F is of the same class for $(x', y') \neq (0, 0)$ and (x, y) in a neighborhood of those on $L_{12} + E_{12}$. Both these functions satisfy the usual homogeneity conditions

$$\begin{aligned} F(x, y, kx', ky') &= kF(x, y, x', y'), \\ G(x, y, kx', ky') &= kG(x, y, x', y') \end{aligned}$$

in these neighborhoods for every $k > 0$.

The necessary conditions that $J(E)$ be a maximum with respect to all curves of \mathfrak{M} with the same end-points, and keeping K fixed, must be fulfilled. Hence we have at once the usual Euler, Weierstrass, Legendre and Jacobi conditions referred to above. These may be stated as follows:

I. *Euler Condition*.—The curve E must satisfy for a certain constant value λ the Euler differential equations

$$H_x - \frac{d}{dt} H_{x'} = 0, \quad H_y - \frac{d}{dt} H_{y'} = 0, \quad (1)$$

where $H = F + \lambda G$. Such curves will, as usual, be called extremals.

II. *Weierstrass Condition*.—The Weierstrass E function

$$E(x, y, p, q, x', y'; \lambda) = H(x, y, x', y'; \lambda) - x'H_{x'}(x, y, p, q; \lambda) - y'H_{y'}(x, y, p, q; \lambda)$$

must be greater than or equal to zero for all $(x, y, p, q; x', y')$, such that (x, y, p, q) belongs to a point of E_{12} while (x', y') is different from $(0, 0)$.

III. *Legendre Condition*.—Along the arc E_{12} , $H_1 \leq 0$, where

$$H_1 = \frac{H_{x'x'}}{y'^2} = -\frac{H_{x'y'}}{x'y'} = \frac{H_{y'y'}}{x'^2}.$$

*An arc is said to be regular when it is continuous and consists of a finite number of arcs each of which has a well-defined and continuously turning tangent.

† See Bolza, *loc. cit.*, p. 13.

IV. Finally E_{12} must satisfy the *Jacobi Condition* for fixed end-points; that is, the extremal arc E_{12} must not contain in its interior either the point P'_1 conjugate to P_1 , or the point P'_2 conjugate to P_2 .

In the transversality condition there is a departure from the result obtained by Bolza* in a closely related problem. We proceed to its determination, however, in an analogous manner. Consider one end-point, say P_1 , fixed. It is possible to set up in the usual way† a one-parameter family of variation curves

$$x = \phi(t, \kappa), \quad y = \psi(t, \kappa), \quad t_1 \leq t \leq t_2,$$

which have the following properties. For $\kappa = \kappa_2$, $t_1 \leq t \leq t_2$, the family contains the arc E_{12} . Furthermore, every arc passes through the point P_1 for $t = t_1$, and intersects the arc L for $t = t_2$, which gives rise to the equations

$$\begin{aligned} \phi(t_1, \kappa) &= x_1, & \psi(t_1, \kappa) &= y_1, \\ \phi(t_2, \kappa) &= x(\kappa), & \psi(t_2, \kappa) &= \tilde{y}(\kappa). \end{aligned}$$

Finally, along each one of these curves, the integral K has the assigned value k . Substituting this family of variation curves in the expression for J we find

$$J(\kappa) = \int_{x_1}^{\kappa_1} F(x, \tilde{y}, x', \tilde{y}') dx + \int_{t_1}^{t_2} F(\phi, \psi, \phi', \psi') dt.$$

Following the procedure of Bolza we have the condition that at the point P_2 ,

$$F(\tilde{x}, \tilde{y}, \tilde{x}', y') - H_{x'}(\phi, \psi, \phi', \psi') \tilde{x}' - H_{y'}(\phi, \psi, \phi', \psi') \tilde{y}' = 0,$$

and by a similar argument at P_1 we have the following result:

V. *Transversality Condition.*—The curve L must cut E_{12} transversally at P_1 and P_2 , that is, at both these points the condition

$$F(\tilde{x}, \tilde{y}, x', \tilde{y}') - H_{x'}(\phi, \psi, \phi', \psi') \tilde{x}' - H_{y'}(\phi, \psi, \phi', \psi') \tilde{y}' = 0 \quad (2)$$

must hold.

§ 2. *Derivatives of the Extremal Integral.*

We suppose now that the arc E_{12} satisfies the necessary conditions of the preceding section, and further that the Legendre and Jacobi conditions for fixed end-points hold in the so-called stronger form. This means that $H_1 < 0$ along the arc E_{12} , and that P_1 and P_2 are not conjugate points on E_{12} . As a result of the continuity properties of F and G , and the fact that $H_1 \neq 0$ along E_{12} , it is known‡ that this arc may be imbedded in a family of extremals

$$x = \phi(t, \alpha, \beta, \lambda), \quad y = \psi(t, \alpha, \beta, \lambda) \quad (3)$$

* *Loc. cit.*, pp. 519, 520.

† Bolza, *loc. cit.*, pp. 473, 474.

‡ Bolza, *loc. cit.*, pp. 468 ff.

which contains E_{12} for values $\alpha_0, \beta_0, \lambda_0, t_1 \leq t \leq t_2$. The functions $\phi, \phi_t, \psi, \psi_t$ are of the class C' in all their arguments in a neighborhood of these values. The constant λ is the isoperimetric constant for each extremal.

Suppose now that M_1 and M_2 are any two points (x_1, y_1) and (x_2, y_2) sufficiently near to P_1 and P_2 , respectively. The equations

$$\left. \begin{aligned} \phi(\tau_1, \alpha, \beta, \lambda) &= x_1, & \phi(\tau_2, \alpha, \beta, \lambda) &= x_2, \\ \psi(\tau_1, \alpha, \beta, \lambda) &= y_1, & \psi(\tau_2, \alpha, \beta, \lambda) &= y_2, \\ \int_{\tau_1}^{\tau_2} G(\phi, \psi, \phi', \psi') dt &= k, \end{aligned} \right\} \quad (4)$$

have the initial solution

$$(\alpha, \beta, \lambda, \tau_1, \tau_2, x_1, y_1, x_2, y_2) = (\alpha_0, \beta_0, \lambda_0, t_1, t_2, x_{10}, y_{10}, x_{20}, y_{20}),$$

where $(x_{10}, y_{10}), (x_{20}, y_{20})$ are now the coordinates of P_1, P_2 , respectively. Furthermore, since the Jacobi condition in its stronger form is satisfied by the arc E_{12} , it follows that the functional determinant of the left members for $\alpha, \beta, \lambda, \tau_1, \tau_2$ is different from zero at this initial solution. It reduces in fact, after suitable transformations, to the determinant $D(t_1, t_2)^*$ formed for E_{12} , which vanishes only when P_1 and P_2 are conjugate. Hence equations (4) have unique solutions of the form

$$\left. \begin{aligned} \alpha(x_1, y_1, x_2, y_2), & \quad \beta(x_1, y_1, x_2, y_2), & \quad \lambda(x_1, y_1, x_2, y_2), \\ \tau_1(x_1, y_1, x_2, y_2), & \quad \tau_2(x_1, y_1, x_2, y_2), \end{aligned} \right\} \quad (5)$$

reducing to $\alpha_0, \beta_0, \lambda_0, t_1, t_2$ for $(x_1, y_1, x_2, y_2) = (x_{10}, y_{10}, x_{20}, y_{20})$, and of class C' near these values. Substituting from these last functions for α, β, λ , in equations (5), we have a family of extremals

$$x = \phi(t, x_1, y_1, x_2, y_2), \quad y = \psi(t, x_1, y_1, x_2, y_2), \quad (6)$$

and two functions, $\tau_1(x_1, y_1, x_2, y_2)$ and $\tau_2(x_1, y_1, x_2, y_2)$ for which the following conditions are then satisfied:

$$\left. \begin{aligned} \phi(\tau_1, x_1, y_1, x_2, y_2) &= x_1, & \phi(\tau_2, x_1, y_1, x_2, y_2) &= x_2, \\ \psi(\tau_1, x_1, y_1, x_2, y_2) &= y_1, & \psi(\tau_2, x_1, y_1, x_2, y_2) &= y_2, \\ \int_{\tau_1}^{\tau_2} G(\phi, \psi, \phi', \psi') dt &= k. \end{aligned} \right\} \quad (7)$$

By differentiating these we find that the following relations hold at the point M_1 :

$$\left. \begin{aligned} \phi' \tau_{1x_1} + \phi_{x_1} &= 1, & \psi' \tau_{1x_1} + \psi_{x_1} &= 0, \\ \phi' \tau_{1y_1} + \phi_{y_1} &= 0, & \psi' \tau_{1y_1} + \psi_{y_1} &= 1, \\ \phi' \tau_{1x_2} + \phi_{x_2} &= 0, & \psi' \tau_{1x_2} + \psi_{x_2} &= 0, \\ \phi' \tau_{1y_2} + \phi_{y_2} &= 0, & \psi' \tau_{1y_2} + \psi_{y_2} &= 0, \end{aligned} \right\} \quad (8)$$

* Bolza, *loc. cit.*, p. 478.

while at the point M_2 ,

$$\left. \begin{aligned} \phi' \tau_{2x_1} + \phi_{x_1} &= 0, & \psi' \tau_{2x_1} + \psi_{x_1} &= 0, \\ \phi' \tau_{2y_1} + \phi_{y_1} &= 0, & \psi' \tau_{2y_1} + \psi_{y_1} &= 0, \\ \phi' \tau_{2x_2} + \phi_{x_2} &= 1, & \psi' \tau_{2x_2} + \psi_{x_2} &= 0, \\ \phi' \tau_{2y_2} + \phi_{y_2} &= 0, & \psi' \tau_{2y_2} + \psi_{y_2} &= 1. \end{aligned} \right\} \quad (9)$$

In accordance with the notation and nomenclature of Bolza,* we use the notation

$$I(x_1, y_1, x_2, y_2) = \int_{\tau_1}^{\tau_2} F(\phi, \psi, \phi', \psi') dt, \quad (10)$$

and call this expression the extremal integral. We desire to obtain the partial derivatives of the function I with respect to its four arguments. In order to simplify the results we make use of two important functions Ω and Ψ , which will now be introduced.

The problem under consideration may be interpreted as a problem in space by defining a third coordinate z by the equation

$$z = \chi(t, x_1, y_1, x_2, y_2) = \int_{\tau_1}^t G(\phi, \psi, \phi', \psi') dt, \quad (11)$$

where ϕ, ψ are of the form given in (6). By differentiation of (11) with respect to t and α , where α is an arbitrarily selected one of the elements x_1, y_1, x_2, y_2 , we obtain

$$-\chi_t + G(\phi, \psi, \phi', \psi') = 0, \quad \chi_\alpha = \int_{\tau_1}^t (G_x \phi_\alpha + G_y \psi_\alpha + G_{x'} \phi'_\alpha + G_{y'} \psi'_\alpha) - G|_{\tau_1} dt. \quad (12)$$

If h is defined by the equation

$$h = F + \lambda(G - z') = H - \lambda z',$$

then the equations

$$h_x - \frac{d}{dt} h_{x'} = 0, \quad h_y - \frac{d}{dt} h_{y'} = 0, \quad h_z - \frac{d}{dt} h_{z'} = 0 \quad (13)$$

are satisfied along any extremal arc, and might be called the Euler equations of the space problem.

We now set up the function Ω by means of the following equation:

$$\begin{aligned} 2\Omega = & \left(\frac{H_{xx} H_{xy}}{H_{yx} H_{yy}} \right) (\xi, \eta) (\xi, \eta) + 2 \left(\frac{H_{x'x} H_{x'y}}{H_{y'x} H_{y'y}} \right) (\xi, \eta) (\xi', \eta') \\ & + \left(\frac{H_{x'x'} H_{x'y'}}{H_{y'x'} H_{y'y'}} \right) (\xi', \eta') (\xi', \eta') + 2\mu (G_x \xi + G_y \eta + G_{x'} \xi' + G_{y'} \eta' - \zeta'), \end{aligned} \quad (14)$$

in which the notation is explained by the equation

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} (x_1, y_1) (x_2, y_2) = (A_{11}x_1 + A_{12}y_1)x_2 + (A_{21}x_1 + A_{22}y_1)y_2.$$

* *Loc. cit.*, pp. 308 ff.

Then the equations analogous to those of Jacobi for other problems in the Calculus of Variations are the following:

$$\left. \begin{aligned} \Omega_{\xi} - \frac{d}{dt} \Omega_{\xi'} &\equiv H_{xx}\xi + H_{xy}\eta + H_{xx'}\xi' + H_{xy'}\eta' + G_x\mu \\ &\quad - \frac{d}{dt} (H_{x'x}\xi + H_{x'y}\eta + H_{x'x'}\xi' + H_{x'y'}\eta' + G_{x'}\mu) = 0, \\ \Omega_{\eta} - \frac{d}{dt} \Omega_{\eta'} &\equiv H_{yx}\xi + H_{yy}\eta + H_{yx'}\xi' + H_{yy'}\eta' + G_y\mu \\ &\quad - \frac{d}{dt} (H_{y'x}\xi + H_{y'y}\eta + H_{y'x'}\xi' + H_{y'y'}\eta' + G_{y'}\mu) = 0, \\ \Omega_{\zeta} - \frac{d}{dt} \Omega_{\zeta'} &\equiv \frac{d}{dt} \mu = 0, \\ \Omega_{\mu} - \frac{d}{dt} \Omega_{\mu'} &\equiv G_x\xi + G_y\eta + G_{x'}\xi' + G_{y'}\eta' - \zeta' = 0. \end{aligned} \right\} \quad (15)$$

These equations are satisfied by the functions

$$(\xi, \eta, \zeta, \mu) = (\phi_{\alpha}, \psi_{\alpha}, \chi_{\alpha}, \lambda_{\alpha}) \quad (16)$$

obtained from (5), (6) and (12), where α stands for any one of x_1, y_1, x_2, y_2 . This is proved by substituting the functions (6) in the Euler equations (1) and differentiating the resulting identities with respect to α , and by differentiating (12) for t .

The expressions for the values of χ_{α} at the points M_1 and M_2 will be useful in later simplifications, and will be computed now. The last equation of (7) is satisfied by the functions (6), and hence by differentiation we obtain

$$\int_{\tau_1}^{\tau_2} (G_x\phi_{\alpha} + G_y\psi_{\alpha} + G_{x'}\phi'_{\alpha} + G_{y'}\psi'_{\alpha}) dt + G|{}^2t_{1\alpha} - G|{}^1t_{1\alpha} = 0,$$

that is, from (12),

$$\chi_{\alpha}(\tau_2, x_1, y_1, x_2, y_2) = -G|{}^1t_{1\alpha}, \quad (17)$$

where it is to be remembered that α is now not one of the constants in (3), but one of the variables x_1, y_1, x_2, y_2 . By direct computation from the expression (12) for χ_{α} , we obtain

$$\chi_{\alpha}(\tau_2, x_1, y_1, x_2, y_2) = -G|{}^1\tau_{1\alpha}. \quad (18)$$

Since Ω is a quadratic form in $\xi, \eta, \zeta, \mu, \xi', \eta', \zeta', \mu'$, we have the relations

$$\begin{aligned} \Sigma(\xi \Omega_{\xi} + \xi' \Omega_{\xi'}) &= 2\Omega, \\ \Sigma(\xi_1 \Omega_{\xi_2} + \xi'_1 \Omega_{\xi'_2}) &= \Sigma(\xi_2 \Omega_{\xi_1} + \xi'_2 \Omega_{\xi'_1}), \end{aligned}$$

where Σ denotes summation over the four elements ξ, η, ζ, μ , and the notation Ω_{ξ_2} , for example, denotes the function obtained by differentiating Ω with

respect to ξ , and then replacing ξ by ξ_2 , η by η_2 , etc. It follows from the second of these that

$$\Sigma \left\{ \xi_1 \left(\Omega_{\xi_2} - \frac{d}{dt} \Omega_{\xi'_2} \right) - \xi_2 \left(\Omega_{\xi_1} - \frac{d}{dt} \Omega_{\xi'_1} \right) \right\} = - \frac{d}{dt} \Psi \left(\xi_1, \eta_1, \zeta_1, \mu_1, \xi'_1, \eta'_1, \zeta'_1, \mu'_1 \right),$$

where

$$\Psi = \Sigma (\xi_1 \Omega_{\xi'_2} - \xi_2 \Omega_{\xi'_1}).$$

We see, therefore, that the function Ψ is a constant if $(\xi_1, \eta_1, \zeta_1, \mu_1)$ and $(\xi_2, \eta_2, \zeta_2, \mu_2)$ are both solutions of equations (15).

We may now proceed to the computation of the derivatives of the extremal integral. Let α, β represent any two (or possibly both the same one) of x_1, y_1, x_2, y_2 . By differentiating (10) we obtain

$$I_\alpha = \int_{\tau_1}^{\tau_2} (F_x \phi_\alpha + F_y \psi_\alpha + F_{x'} \phi'_\alpha + F_{y'} \psi'_\alpha) dt + F \tau_\alpha |_1^2. \quad (19)$$

From the isoperimetric condition we have

$$0 = \lambda \int_{\tau_1}^{\tau_2} (G_x \phi_\alpha + G_y \psi_\alpha + G_{x'} \phi'_\alpha + G_{y'} \psi'_\alpha) dt + \lambda G \tau_\alpha |_1^2.$$

Then by adding, performing the Lagrangian partial differentiation, applying the Euler equation (1), and using the well-known homogeneity relation

$$H = x' H_{x'} + y' H_{y'},$$

we obtain

$$I_\alpha = H_{x'} (\phi' \tau_\alpha + \phi_\alpha) + H_{y'} (\psi' \tau_\alpha + \psi_\alpha) |_1^2. \quad (20)$$

Similarly,

$$I_\beta = H_{x'} (\phi' \tau_\beta + \phi_\beta) + H_{y'} (\psi' \tau_\beta + \psi_\beta) |_1^2. \quad (21)$$

Differentiating I_α with respect to β we have

$$\begin{aligned} I_{\alpha\beta} = & (\phi' \tau_\alpha + \phi_\alpha) \left[\tau_\beta \frac{d}{dt} H_{x'} + \Omega_{\xi'} (\phi_\beta, \psi_\beta, \chi_\beta, \lambda_\beta) \right] \\ & + (\psi' \tau_\alpha + \psi_\alpha) \left[\tau_\beta \frac{d}{dt} H_{y'} + \Omega_{\eta'} (\phi_\beta, \psi_\beta, \chi_\beta, \lambda_\beta) \right] \end{aligned} \quad (22)$$

since the values of $\phi' \tau_\alpha + \phi_\alpha$ and $\psi' \tau_\alpha + \psi_\alpha$ at the end-points are independent of x_1, y_1, x_2, y_2 , in every case, as follows from equations (8) and (9). The arguments indicated in the derivatives of Ω are substituted for ξ, η, ζ, μ . Similarly, we obtain

$$\begin{aligned} I_{\beta\alpha} = & (\phi' \tau_\beta + \phi_\beta) \left[\tau_\alpha \frac{d}{dt} H_{x'} + \Omega_{\xi'} (\phi_\alpha, \psi_\alpha, \chi_\alpha, \lambda_\alpha) \right] \\ & + (\psi' \tau_\beta + \psi_\beta) \left[\tau_\alpha \frac{d}{dt} H_{y'} + \Omega_{\eta'} (\phi_\alpha, \psi_\alpha, \chi_\alpha, \lambda_\alpha) \right] |_1^2. \end{aligned} \quad (23)$$

To verify the computations, we prove the equality of these expressions for $I_{\alpha\beta}$ and $I_{\beta\alpha}$. By applying the Euler equations (1), and the relations

$$\begin{aligned} H_{xx'}\phi' + H_{xy'}\psi' &= H_x, & H_{yx'}\phi' + H_{yy'}\psi' &= H_y, \\ H_{x'x'}\phi' + H_{x'y'}\psi' &= 0, & H_{y'x'}\phi' + H_{y'y'}\psi' &= 0, \\ G_{x'}\phi' + G_{y'}\psi' &= G, \end{aligned}$$

all of which arise from the homogeneity conditions, we obtain the following:

$$\left. \begin{aligned} I_{\alpha\beta} &= \phi_a \Omega_{\xi'}(\phi_\beta, \psi_\beta, \chi_\beta, \lambda_\beta) + \psi_a \Omega_{\eta'}(\phi_\beta, \psi_\beta, \chi_\beta, \lambda_\beta) \\ &\quad + \tau_a [H_x \phi_\beta + H_y \psi_\beta + G \lambda_\beta] + (\phi' \tau_a + \phi_a) \tau_\beta H_x + (\psi' \tau_a + \psi_a) \tau_\beta H_y \Big|_1^2, \\ I_{\beta\alpha} &= \phi_\beta \Omega_{\xi'}(\phi_a, \psi_a, \chi_a, \lambda_a) + \psi_\beta \Omega_{\eta'}(\phi_a, \psi_a, \chi_a, \lambda_a) \\ &\quad + \tau_\beta [H_x \phi_a + H_y \psi_a + G \lambda_a] + (\phi' \tau_\beta + \phi_\beta) \tau_a H_x + (\psi' \tau_\beta + \psi_\beta) \tau_a H_y \Big|_1^2. \end{aligned} \right\} \quad (24)$$

The notation $\Omega_{\xi'}$, $\Omega_{\eta'}$ is explained specifically in equations (15), reference to which shows that χ_β , χ_a do not actually occur. By the third equation of (15) together with (16), (17) and (18) we have

$$\begin{aligned} G \tau_\beta \lambda_a \Big|_1^2 &= \chi_\beta \Omega_{\xi'}(\phi_a, \psi_a, \chi_a, \lambda_a) \Big|_1^2, \\ G \tau_a \lambda_\beta \Big|_1^2 &= \chi_a \Omega_{\xi'}(\phi_\beta, \psi_\beta, \chi_\beta, \lambda_\beta) \Big|_1^2. \end{aligned}$$

If we make substitutions accordingly in the above expressions for $I_{\alpha\beta}$ and $I_{\beta\alpha}$, and form their difference, we have

$$\begin{aligned} I_{\alpha\beta} - I_{\beta\alpha} &= \phi_a \Omega_{\xi'}(\phi_\beta, \psi_\beta, \chi_\beta, \lambda_\beta) - \phi_\beta \Omega_{\xi'}(\phi_a, \psi_a, \chi_a, \lambda_a) \\ &\quad + \psi_a \Omega_{\eta'}(\phi_\beta, \psi_\beta, \chi_\beta, \lambda_\beta) - \psi_\beta \Omega_{\eta'}(\phi_a, \psi_a, \chi_a, \lambda_a) \\ &\quad + \chi_a \Omega_{\xi'}(\phi_\beta, \psi_\beta, \chi_\beta, \lambda_\beta) - \chi_\beta \Omega_{\xi'}(\phi_a, \psi_a, \chi_a, \lambda_a) \\ &= \Psi \left(\phi_a, \psi_a, \chi_a, \lambda_a, \phi'_a, \psi'_a, \chi'_a, \lambda'_a \right. \\ &\quad \left. \phi_\beta, \psi_\beta, \chi_\beta, \lambda_\beta, \phi'_\beta, \psi'_\beta, \chi'_\beta, \lambda'_\beta \right) \Big|_1^2. \end{aligned}$$

But the two sets of arguments in Ψ satisfy equations (15), and hence the function Ψ in the last equation is independent of t . Its values at $t=\tau_1$ and $t=\tau_2$ are the same, and hence it follows that $I_{\alpha\beta} = I_{\beta\alpha}$.

The desired partial derivatives of the extremal integral may now be computed very easily. In the expression (20) for I_a let α take successively the values x_1, y_1, x_2, y_2 . Making use of relations (8) and (9) we obtain

$$I_{x_1} = -H_{x'} \Big|_1^1, \quad I_{y_1} = -H_{y'} \Big|_1^1, \quad I_{x_2} = +H_{x'} \Big|_2^2, \quad I_{y_2} = +H_{y'} \Big|_2^2. \quad (27)$$

Of the second derivatives ten are required, viz., $I_{x_1x_1}, I_{x_1y_1}, I_{y_1y_1}, I_{x_1y_2}, I_{y_1x_2}, I_{y_1y_2}, I_{x_2x_2}, I_{x_2y_2}, I_{y_2y_2}$. With the exception of the cases in which the two subscripts are the same, the derivatives occur in two different forms, whose values, however, are of course the same. The following results are obtained most readily from (22) and (23). We employ relations (8) and (9) and the Euler equations (1).

$$\begin{aligned}
I_{x_1x_1} &= -[H_x \tau_{x_1} + \Omega_{\xi'}(\phi_{x_1}, \psi_{x_1}, \chi_{x_1}, \lambda_{x_1})]^1, \\
I_{x_1y_1} &= -[H_x \tau_{y_1} + \Omega_{\xi'}(\phi_{y_1}, \psi_{y_1}, \chi_{y_1}, \lambda_{y_1})]^1 = -[H_y \tau_{x_1} + \Omega_{\eta'}(\phi_{x_1}, \psi_{x_1}, \chi_{x_1}, \lambda_{x_1})]^1, \\
I_{y_1y_1} &= -[H_y \tau_{y_1} + \Omega_{\eta'}(\phi_{y_1}, \psi_{y_1}, \chi_{y_1}, \lambda_{y_1})]^1, \\
I_{x_1x_2} &= -[H_x \tau_{x_2} + \Omega_{\xi'}(\phi_{x_2}, \psi_{x_2}, \chi_{x_2}, \lambda_{x_2})]^1 = +[H_x \tau_{x_1} + \Omega_{\xi'}(\phi_{x_1}, \psi_{x_1}, \chi_{x_1}, \lambda_{x_1})]^2, \\
I_{x_1y_2} &= -[H_x \tau_{y_2} + \Omega_{\xi'}(\phi_{y_2}, \psi_{y_2}, \chi_{y_2}, \lambda_{y_2})]^1 = +[H_y \tau_{x_1} + \Omega_{\eta'}(\phi_{x_1}, \psi_{x_1}, \chi_{x_1}, \lambda_{x_1})]^2, \\
I_{y_1x_2} &= -[H_y \tau_{x_2} + \Omega_{\eta'}(\phi_{x_2}, \psi_{x_2}, \chi_{x_2}, \lambda_{x_2})]^1 = +[H_x \tau_{y_1} + \Omega_{\xi'}(\phi_{y_1}, \psi_{y_1}, \chi_{y_1}, \lambda_{y_1})]^2, \\
I_{y_1y_2} &= -[H_y \tau_{y_2} + \Omega_{\eta'}(\phi_{y_2}, \psi_{y_2}, \chi_{y_2}, \lambda_{y_2})]^1 = +[H_y \tau_{y_1} + \Omega_{\eta'}(\phi_{y_1}, \psi_{y_1}, \chi_{y_1}, \lambda_{y_1})]^2, \\
I_{x_2x_2} &= +[H_x \tau_{x_2} + \Omega_{\xi'}(\phi_{x_2}, \psi_{x_2}, \chi_{x_2}, \lambda_{x_2})]^2, \\
I_{x_2y_2} &= +[H_x \tau_{y_2} + \Omega_{\xi'}(\phi_{y_2}, \psi_{y_2}, \chi_{y_2}, \lambda_{y_2})]^2 = +[H_y \tau_{x_2} + \Omega_{\eta'}(\phi_{x_2}, \psi_{x_2}, \chi_{x_2}, \lambda_{x_2})]^2, \\
I_{y_2y_2} &= +[H_y \tau_{y_2} + \Omega_{\eta'}(\phi_{y_2}, \psi_{y_2}, \chi_{y_2}, \lambda_{y_2})]^2.
\end{aligned}$$

The derivatives $\Omega_{\xi'}$, $\Omega_{\eta'}$ are given by the formulas

$$\begin{aligned}
\Omega_{\xi'}(\xi, \eta, \zeta, \mu) &= H_{x'x}\xi + H_{x'y}\eta + H_{x'x'}\xi' + H_{x'y'}\eta' + G_{x'}\mu, \\
\Omega_{\eta'}(\xi, \eta, \zeta, \mu) &= H_{y'x}\xi + H_{y'y}\eta + H_{y'x'}\xi' + H_{y'y'}\eta' + G_{y'}\mu.
\end{aligned}$$

§ 3. A New Necessary Condition.

Suppose now that arc E_{12} is an extremal satisfying the strengthened Legendre and Jacobi conditions and cut transversally by the arc L at P_1 and P_2 . Form the function

$$J(u, v) = \int_v^u F(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}') d\pi + I(\tilde{x}(u), \tilde{y}(u), \tilde{x}(v), \tilde{y}(v)), \quad (29)$$

where the $\tilde{x}(\pi)$, $\tilde{y}(\pi)$ are those defining the arc L given in Section 1. This function $J(u, v)$ must be a maximum at $(u, v) = (\pi_1, \pi_2)$ if arc E_{12} is to maximize J among arcs with end-points on L which give to integral K the value k . The functions $\alpha, \beta, \lambda, \tau_1, \tau_2$ in (5) become functions of u and v when x_1, y_1, x_2, y_2 are replaced by $\tilde{x}(u), y(u), \tilde{x}(v), \tilde{y}(v)$, so that the family of extremals (3) or (6) has the form

$$x = \phi(t, u, v), \quad y = \psi(t, u, v). \quad (30)$$

Thus we have the relations

$$\left. \begin{aligned}
\phi(t, x_1, y_1, x_2, y_2) &= \phi(t, u, v), & \psi(t, x_1, y_1, x_2, y_2) &= \psi(t, u, v), \\
\tau_1(x_1, y_1, x_2, y_2) &= \tau_1(u, v), & \tau_2(x_1, y_1, x_2, y_2) &= \tau_2(u, v), \\
\lambda(x_1, y_1, x_2, y_2) &= \lambda(u, v).
\end{aligned} \right\} \quad (31)$$

Then by differentiation we obtain

$$\left. \begin{aligned}
\phi_u &= \phi_{x_1} \tilde{x}_u + \phi_{y_1} \tilde{y}_u, & \phi_v &= \phi_{x_2} x_v + \phi_{y_2} \tilde{y}_v, \\
\psi_u &= \psi_{x_1} \tilde{x}_u + \psi_{y_1} \tilde{y}_u, & \psi_v &= \psi_{x_2} \tilde{x}_v + \psi_{y_2} \tilde{y}_v, \\
\tau_{1u} &= \tau_{1x_1} \tilde{x}_u + \tau_{1y_1} \tilde{y}_u, & \tau_{1v} &= \tau_{1x_2} \tilde{x}_v + \tau_{1y_2} \tilde{y}_v, \\
\tau_{2u} &= \tau_{2x_1} \tilde{x}_u + \tau_{2y_1} \tilde{y}_u, & \tau_{2v} &= \tau_{2x_2} \tilde{x}_v + \tau_{2y_2} \tilde{y}_v, \\
\lambda_u &= \lambda_{x_1} \tilde{x}_u + \lambda_{y_1} \tilde{y}_u, & \lambda_v &= \lambda_{x_2} \tilde{x}_v + \lambda_{y_2} \tilde{y}_v.
\end{aligned} \right\} \quad (32)$$

It is evident that

$$\begin{aligned}\Omega_{\xi'}(\phi_u, \psi_u, \chi_u, \lambda_u) &= x_u \Omega_{\xi'}(\phi_{x_1}, \psi_{x_1}, \chi_{x_1}, \lambda_{x_1}) + \tilde{y}_u \Omega_{\xi'}(\phi_{y_1}, \psi_{y_1}, \chi_{y_1}, \lambda_{y_1}), \\ \Omega_{\eta'}(\phi_u, \psi_u, \chi_u, \lambda_u) &= \tilde{x}_u \Omega_{\eta'}(\phi_{x_1}, \psi_{x_1}, \chi_{x_1}, \lambda_{x_1}) + \tilde{y}_u \Omega_{\eta'}(\phi_{y_1}, \psi_{y_1}, \chi_{y_1}, \lambda_{y_1}), \\ \Omega_{\xi'}(\phi_v, \psi_v, \chi_v, \lambda_v) &= \tilde{x}_v \Omega_{\xi'}(\phi_{x_2}, \psi_{x_2}, \chi_{x_2}, \lambda_{x_2}) + \tilde{y}_v \Omega_{\xi'}(\phi_{y_2}, \psi_{y_2}, \chi_{y_2}, \lambda_{y_2}), \\ \Omega_{\eta'}(\phi_v, \psi_v, \chi_v, \lambda_v) &= \tilde{x}_v \Omega_{\eta'}(\phi_{x_2}, \psi_{x_2}, \chi_{x_2}, \lambda_{x_2}) + \tilde{y}_v \Omega_{\eta'}(\phi_{y_2}, \psi_{y_2}, \chi_{y_2}, \lambda_{y_2}).\end{aligned}$$

From (29) by differentiating we obtain

$$\begin{aligned}J_u(u, v) &= F(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')| + I_{x_1} \tilde{x}_u + I_{y_1} \tilde{y}_u, \\ J_v(u, v) &= -F(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')|^2 + I_{x_2} \tilde{x}_v + I_{y_2} \tilde{y}_v, \\ J_{uu}(u, v) &= \frac{d}{du} [F(x, y, \tilde{x}', \tilde{y}')| + I_{x_1} \tilde{x}_{uu} + I_{y_1} \tilde{y}_{uu} \\ &\quad + \tilde{x}_u (I_{x_1 x_1} \tilde{x}_u + I_{x_1 y_1} \tilde{y}_u) + \tilde{y}_u (I_{y_1 x_1} \tilde{x}_u + I_{y_1 y_1} \tilde{y}_u), \\ J_{uv}(u, v) &= \tilde{x}_u (I_{x_1 x_2} \tilde{x}_v + I_{x_1 y_2} \tilde{y}_v) + \tilde{y}_u (I_{y_1 x_2} \tilde{x}_v + I_{y_1 y_2} \tilde{y}_v), \\ J_{vv}(u, v) &= \frac{d}{dv} [-F(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')|^2 + I_{x_2} \tilde{x}_{vv} + I_{y_2} \tilde{y}_{vv} \\ &\quad + \tilde{x}_v (I_{x_2 x_2} \tilde{x}_v + I_{x_2 y_2} \tilde{y}_v) + \tilde{y}_v (I_{y_2 x_2} \tilde{x}_v + I_{y_2 y_2} \tilde{y}_v)].\end{aligned}$$

Substituting in these from (28), performing the indicated operations and making use of relations (32) and (33), we have the following:*

$$\left. \begin{aligned}J_u(u, v) &= F(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}') - \tilde{x}_u H_{x'} - \tilde{y}_u H_{y'}|^1, \\ J_v(u, v) &= -F(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}') + \tilde{x}_v H_{x'} + \tilde{y}_v H_{y'}|^2, \\ J_{uu}(u, v) &= \tilde{x}_{uu} A + \tilde{y}_{uu} B + F_x(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}') \tilde{x}_u + F_y(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}') \tilde{y}_u \\ &\quad - \tilde{x}_u \Omega_{\xi'_1} - \tilde{y}_u \Omega_{\eta'_1} - H_{x'} t_u \tilde{x}_u - H_{y'} t_u \tilde{y}_u|^1, \\ J_{uv}(u, v) &= -\tilde{x}_u \Omega_{\xi'_2} - \tilde{y}_u \Omega_{\eta'_2} - H_{x'} t_v \tilde{x}_u - H_{y'} t_v \tilde{y}_u|^1, \\ &= x_v \Omega_{\xi'_1} + \tilde{y}_v \Omega_{\eta'_1} + H_{x'} t_u \tilde{x}_v + H_{y'} t_u \tilde{y}_v|^2, \\ J_{vv}(u, v) &= -\tilde{x}_v A - \tilde{y}_v B - F_x(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}') \tilde{x}_v - F_y(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}') \tilde{y}_v \\ &\quad + \tilde{x}_v \Omega_{\xi'_2} + \tilde{y}_v \Omega_{\eta'_2} + H_{x'} t_v \tilde{x}_v + H_{y'} t_v \tilde{y}_v|^2,\end{aligned} \right\} \quad (34)$$

where

$$A = F_{x'}(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}') - H_{x'}, \quad B = F_{y'}(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}') - H_{y'}, \quad (35)$$

and the subscript 1 of ξ' and η' denotes that the arguments of $\Omega_{\xi'}$, $\Omega_{\eta'}$ are $\phi_u, \psi_u, \phi'_u, \psi'_u, \lambda_u$, while the subscript 2 denotes that the arguments are $\phi_v, \psi_v, \phi'_v, \psi'_v, \lambda_v$.

Since E_{12} is cut transversally by L , it follows from (2) that when $(u, v) = (x_1, x_2)$,

$$J_u(x_1, x_2) = J_v(x_1, x_2) = 0.$$

* When the arguments are omitted in F, G, H and their derivatives, they are understood to be ϕ, ψ, ϕ', ψ' .

The former of these two equalities may be written in the form

$$[F_{x'}(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}') - H_{x'}]\tilde{x}_u + [F_{y'}(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}') - H_{y'}]\tilde{y}_u = 0,$$

or

$$A\tilde{x}_u + B\tilde{y}_u = 0,$$

where A and B are the functions defined by (35). Then there exists a function m_1 such that at the point P_1 the following relations hold:

$$A|_1 = \frac{-m_1\tilde{y}_u}{(\sqrt{\tilde{x}_u^2 + \tilde{y}_u^2})^3}, \quad B|_1 = \frac{m_1\tilde{x}_u}{(\sqrt{\tilde{x}_u^2 + \tilde{y}_u^2})^3}. \quad (36)$$

By a similar argument we have at the point P_2 the relations

$$A|_2 = \frac{-m_2\tilde{y}_v}{(\sqrt{\tilde{x}_v^2 + \tilde{y}_v^2})^3}, \quad B|_2 = \frac{m_2\tilde{x}_v}{(\sqrt{\tilde{x}_v^2 + \tilde{y}_v^2})^3}. \quad (37)$$

By substitution from (36) and (37) it follows that J_{uu} , J_{uv} , J_{vv} have the forms

$$J_{uu}(u, v) = \frac{m_1}{r_1} + R_1, \quad J_{uv}(u, v) = S_1 = -S_2, \quad J_{vv}(u, v) = -\frac{m_2}{r_2} - R_2,$$

where

$$m_1 = (A^2 + B^2)|_1 \sqrt{\tilde{x}_u^2 + \tilde{y}_u^2}, \quad m_2 = (A^2 + B^2)|_2 \sqrt{\tilde{x}_v^2 + \tilde{y}_v^2}, \quad (38)$$

$$\left. \begin{aligned} R_1 &= F_x(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')\tilde{x}_u + F_y(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')\tilde{y}_u \\ &\quad - \tilde{x}_u\Omega_{\xi'_1} - \tilde{y}_u\Omega_{\eta'_1} - H_x t_u \tilde{x}_u - H_y t_u \tilde{y}_u|_1, \\ R_2 &= F_x(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')\tilde{x}_v + F_y(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')\tilde{y}_v \\ &\quad - \tilde{x}_v\Omega_{\xi'_2} - \tilde{y}_v\Omega_{\eta'_2} - H_x t_v \tilde{x}_v - H_y t_v \tilde{y}_v|_2, \end{aligned} \right\} \quad (39)$$

$$\left. \begin{aligned} S_1 &= -x_u\Omega_{\xi'_2} - \tilde{y}_u\Omega_{\eta'_2} - H_x t_v \tilde{x}_u - H_y t_v \tilde{y}_u|_1, \\ S_2 &= -\tilde{x}_v\Omega_{\xi'_1} - \tilde{y}_v\Omega_{\eta'_1} - H_x t_u \tilde{x}_v - H_y t_u \tilde{y}_v|_2, \end{aligned} \right\} \quad (40)$$

and r_1 and r_2 are the radii of curvature of L at the points P_1 and P_2 , respectively.

If arc E_{12} is to give to the function $J(u, v)$ a maximum value, it is necessary not only that all preceding conditions be satisfied, but also that the second partial derivatives of $J(u, v)$ satisfy the following conditions:

$$J_{uu}(u, v) \leq 0, \quad J_{vv}(u, v) \leq 0, \quad J_{uu}(u, v)J_{vv}(u, v) - J_{uv}^2(u, v) \geq 0.$$

Thus we have a new condition for the problem, analogous to the Jacobi condition in other problems in the Calculus of Variations. We may summarize our results as follows:

Assumed that the Euler, Weierstrass, Legendre and Jacobi conditions are satisfied for the corresponding problem with fixed end-points—the Legendre and Jacobi in the stronger form—and that the arc E_{12} is cut transversally by

the fixed curve L , then a further necessary condition that arc E shall furnish a maximum for the problem at hand is that

$$\left. \begin{aligned} \text{(a)} \quad & \frac{m_1}{r_1} + R_1 \leq 0, \quad -\frac{m_2}{r_2} - R_2 \leq 0, \\ \text{(b)} \quad & -\frac{m_1 m_2}{r_1 r_2} - \frac{m_1 R_2}{r_1} - \frac{m_2 R_1}{r_2} - R_1 R_2 + S_1 S_2 \geq 0, \end{aligned} \right\} \quad (41)$$

where the notation is explained by equations (38), (39) and (40).

§ 4. *Sufficient Conditions.*

In the determination of conditions which are sufficient, direct application is made of a theorem proved by Hahn.* This theorem holds for the general Lagrangian problem, where certain isoperimetric conditions are to be fulfilled while the end-points of the comparison arcs are to satisfy any number of pre-assigned conditions. In so far as it relates to the problem at hand, this theorem may be formulated as follows:

Let E_{12} be an extremal arc

$$x = \phi(t), \quad y = \psi(t), \quad t_1 \leq t \leq t_2,$$

satisfying the Euler, Weierstrass, Legendre and Jacobi conditions,† the last three in the stronger forms. Then there exist weak neighborhoods $(E_{12})'_\rho$, $(E_{12})'_\sigma$, $\sigma \leq \rho$, such that every extremal arc E_{34} in $(E_{12})'_\sigma$ furnishes a maximum $I(E_{34})$ for the integral

$$I = \int_{\tau_1}^{\tau_2} F(x, y, x', y') dt$$

with respect to all arcs V_{34} in $(E_{12})'_\rho$ such that $K(V_{34}) = K(E_{34})$.

Let E_{34} be an extremal arc joining a point $P_3(x=\kappa_3)$ to a point $P_4(x=\kappa_4)$ on L and having $K(E_{34}) = K(E_{12})$. The extremals E_{34} form a two-parameter family with parameters κ_3, κ_4 , containing E_{12} for $\kappa_3 = \kappa_1, \kappa_4 = \kappa_2$. If the conditions given in the theorem of § 3 are changed by the substitution, in the place of (41), of the conditions

$$\left. \begin{aligned} \text{(a)} \quad & \frac{m_1}{r_1} - R_1 < 0, \quad -\frac{m_1 m_2}{r_1 r_2} - R_2 < 0, \\ \text{(b)} \quad & \frac{m_1 m_2}{r_1 r_2} - \frac{m_1 R_2}{r_1} - \frac{m_2 R_1}{r_2} - R_1 R_2 + S_1 S_2 > 0, \end{aligned} \right\} \quad (41')$$

* See Hahn, "Ueber Variations Probleme mit variablen Endpunkten," *Monatshefte für Mathematik und Physik*, Vol. XXII (1911), p. 127.

† These are conditions I, II', III', IV' of Bolza, *loc. cit.*, p. 514 with the proper changes for the determination of a maximum instead of a minimum.

we have a set of conditions which are sufficient to insure that E_{12} furnishes a maximum among the curves E_{34} ; i. e., that

$$J(E_{12}) > J(E_{34}), \quad E_{34} \neq E_{12}.$$

Now if $\tau \leq \sigma$ is sufficiently small, every arc V_{34} with $K(V_{34}) = K(E_{12})$ in $(E_{12})'_\tau$ determines an extremal E_{34} in $(E_{12})'_\sigma$ of the Hahn theorem with $K(E_{34}) = K(V_{34}) = K(E_{12})$, and so near E_{12} that

$$J(E_{34}) < J(E_{12}),$$

according to the preceding paragraph. If then E_{12} satisfies in addition to the above conditions the Weierstrass condition in the stronger form, the hypothesis of the Hahn theorem is satisfied and consequently we have

$$J(V_{34}) < J(E_{34}).$$

Hence it follows that

$$J(V_{34}) < J(E_{12})$$

for all variation arcs V_{34} with $K(V_{34}) = K(E_{12})$ in $(E_{12})'_\tau$. We have thus proved the theorem:

If E_{12} satisfies the Euler condition, the transversality condition, and (41'), together with the stronger forms of the Weierstrass, Legendre and Jacobi conditions, then there exists a τ such that E_{12} furnishes a maximum $J(E_{12})$ for the function

$$J = \int_{L_{21}} F(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}') dx + \int_{E_{12}} F(x, y, x', y') \cdot dt$$

with respect to all arcs V_{34} in $(E_{12})'_\tau$ with $K(V_{34}) = K(E_{12})$.

§5. Geometric Interpretations.

The conditions found in Section 4 may be interpreted geometrically by the use of a set of oblique axes in the plane. The coordinates of points in the plane referred to this set of axes will be considered as possible values of radii of curvature of the curve L at the points P_1 and P_2 . Critical points are those at which the equalities

$$\left. \begin{aligned} \text{(a)} \quad r_1 + \frac{m_1}{R_1} &= 0, \quad r_2 + \frac{m_2}{R_2} = 0, \\ \text{(b)} \quad r_1 r_2 + \frac{m_2 R_1 r_1}{R_1 R_2 + S_1^2} + \frac{m_1 R_2 r_2}{R_1 R_2 + S_1^2} + \frac{m_1 m_2}{R_1 R_2 + S_1^2} &= 0, \end{aligned} \right\} \quad (42)$$

are satisfied, special discussion being required when either r_1 or r_2 vanishes. In any particular case the signs of the various functions are of course

determined. We limit this discussion of the general problem to the case for which $m_1 > 0, R_1 < 0, m_2 < 0, R_2 > 0, c = R_1 R_2 + S_1^2 < 0$,

and will consider that in the given situation, only r_1 and r_2 may vary.

We suppose that in the given situation, the extremal arc E_{12} (see Fig. 3) is cut transversally by the fixed curve L at P_1 and P_2 . Through these two points draw lines $X'_1 X_2$ and $X'_2 X_2$ parallel to r_1 and r_2 respectively, and intersecting* at O . We take these lines as a set of oblique axes, OX_1 and OX_2 being the positive directions. The first two equalities of (42) determine two straight lines $M'M$ and $N'N$ parallel to the axes. The last equality of (42) is

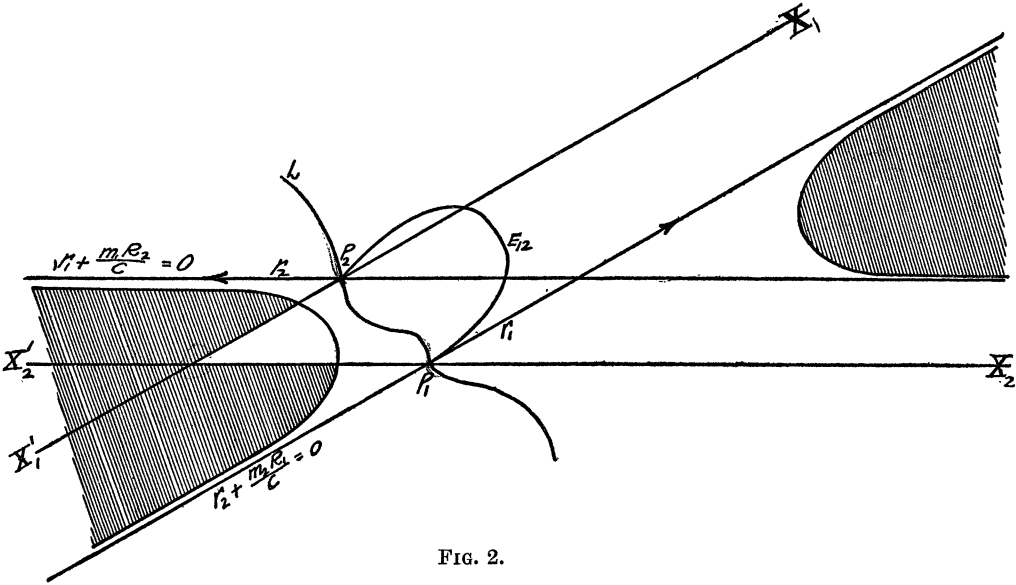


FIG. 2.

an hyperbola lying in the first and third quadrants and asymptotic to the two lines

$$r_1 + \frac{m_1 R_2}{c} = 0, \quad r_2 + \frac{m_2 R_1}{c} = 0.$$

Reference to condition (41) then shows that

In order that two quantities ρ_1, ρ_2 , be suitable values for r_1 and r_2 for the existence of a maximum, it is necessary and sufficient that the point (ρ_1, ρ_2) lie in the portion of the plane shaded in Fig. 2.

§ 6. *Application to the Problem of Dido.*

Let L be a given fixed curve. We wish to determine an arc E_{12} with end-points P_1 and P_2 on L and with a given length k , such that the area enclosed

*In case these lines do not intersect, some other set of lines may be used; for instance, line r_1 and the line perpendicular to it through P .

by the arc P_2P_1 of L and the arc P_1P_2 of E shall be a maximum, P_1 and P_2 to be distinct points.

The functions F and G for this problem are

$$F(x, y, x', y') = \frac{1}{2}(xy' - yx'), \quad G(x, y, x', y') = \sqrt{x'^2 + y'^2}.$$

Suppose that the equations of the fixed curve L are

$$x = \tilde{x}(\kappa), \quad y = \tilde{y}(\kappa), \quad (L)$$

and denote by u and v the values of κ in the neighborhoods of the values of κ_1 and κ_2 , which latter values define P_1 and P_2 , respectively. Let the equation of E_{12} be

$$x = \phi(t), \quad y = \psi(t), \quad t_1 \leq t \leq t_2. \quad (E)$$

We have then to maximize the function

$$J = \frac{1}{2} \int_v^u (\tilde{x}\tilde{y}' - \tilde{y}\tilde{x}') dx + \frac{1}{2} \int_{t_1}^{t_2} (\phi\psi' - \psi\phi') dt,$$

while the integral K :

$$K = \int_{t_1}^{t_2} \sqrt{\phi'^2 + \psi'^2} dt$$

is to have a preassigned value k .

Applying the results of §§ 1, 3, 4, we obtain the following conditions for the problem:

I. E_{12} is the arc of a circle *

$$x = \alpha - \lambda \cos t, \quad y = \beta - \lambda \sin t, \quad t_1 \leq t \leq t_2.$$

$$\text{II. } E(x, y, p, q, x', y'; \lambda) = \lambda \frac{(px' - qy')^2}{\sqrt{p^2 + q^2} \{ \sqrt{p^2 + q^2} \sqrt{x'^2 + y'^2} + px' + qy' \}} \leq 0,$$

i. e., $\lambda \leq 0$ for (x, y, p, q) on E_{12} for every $(x', y') \neq (0, 0)$. Since by condition I, λ is seen to be the radius of the circular arc E_{12} , the value zero is necessarily excluded, and we have the stronger condition $\lambda < 0$.

III. $H_1 = \frac{\lambda}{(\sqrt{x'^2 + y'^2})^3} \leq 0$ along E_{12} . This condition follows directly from condition II, and in fact in the stronger form.

IV. E_{12} contains no conjugate point to P_1 or P_2 ; that is, since P_1 and P_2 are distinct, $t_2 < t_1 + 2\pi$.

$$\text{Now from I,} \quad \sqrt{\phi'^2 + \psi'^2} = -\lambda.$$

Since E and L intersect at P_1 we have

$$|\phi'| = \lambda \sin t_1, \quad |\psi'| = -\lambda \cos t_1.$$

* As to the determination of conditions I-V, cf. also Bolza, *loc. cit.*, pp. 465, 483.

Accordingly we have for the transversality condition,

$$\tilde{x}' \sin t_1 - \tilde{y}' \cos t_1 = 0$$

and a similar result at P_2 . We therefore have the condition

V. E_{12} cuts the curve L orthogonally at both P_1 and P_2 .

If now the length of arc is chosen as the parameter κ , we have from condition V the important relations:

$$\begin{aligned}\tilde{x}_u(u) &= \cos t_1, & \tilde{y}_u(u) &= \sin t_1, \\ \tilde{x}_v(v) &= -\cos t_2, & \tilde{y}_v(v) &= -\sin t_2.\end{aligned}$$

Employing these relations and the values of $\alpha_u, \beta_u, \alpha_v, \beta_v$ from the Euler equations, we obtain from the last three expressions of (34) the following expressions for the second partial derivatives:

$$J_{uu}(u, v) = -\frac{\lambda}{r_1} - \frac{D_2}{D_1}, \quad J_{uv}(u, v) = \frac{D_3}{D_1}, \quad J_{vv}(u, v) = -\frac{\lambda}{r_2} - \frac{D_2}{D_1},$$

where

$$D_1 = 2 - 2 \cos(t_2 - t_1) - (t_2 - t_1) \sin(t_2 - t_1),$$

$$D_2 = -\sin(t_2 - t_1) + (t_2 - t_1) \cos(t_2 - t_1),$$

$$D_3 = (t_2 - t_1) - \sin(t_2 - t_1),$$

$$r_1 = \frac{1}{|\tilde{x}_u \tilde{y}_{uu} - \tilde{y}_u \tilde{x}_{uu}|}, \quad r_2 = \frac{1}{|x_v \tilde{y}_{vv} - \tilde{y}_v \tilde{x}_{vv}|},$$

r_1 and r_2 being the radii of curvature of the curve L at P_1 and P_2 , respectively.

Conditions (41) are therefore *

$$-\frac{\lambda}{r_1} - \frac{D_2}{D_1} \leq 0, \quad -\frac{\lambda}{r_2} - \frac{D_2}{D_1} \leq 0, \quad \left[-\frac{\lambda}{r_1} - \frac{D_2}{D_1}\right] \cdot \left[-\frac{\lambda}{r_2} - \frac{D_2}{D_1}\right] - \frac{D_3^2}{D_1^2} \geq 0.$$

It is evident that if the first and third of these conditions are satisfied, the second must also hold.

We may state the result as follows:

In order that E_{12} and L_{21} enclose a maximum area in the Problem of Dido stated above, it is necessary that E_{12} be a circle-arc

$$x = \alpha - \lambda \cos t, \quad y = \beta - \lambda \sin t, \quad t_1 \leq t \leq t_2,$$

with $\lambda < 0$ and $t_2 < t_1 + 2\pi$, cutting L orthogonally at P_1 and P_2 , and that the conditions

$$-\frac{\lambda}{r_1} - \frac{D_2}{D_1} \leq 0, \quad \left(-\frac{\lambda}{r_1} - \frac{D_2}{D_1}\right) \left(-\frac{\lambda}{r_2} - \frac{D_2}{D_1}\right) - \frac{D_3^2}{D_1^2} \geq 0$$

be fulfilled.

Applying the results of § 4, we have immediately the following:

* With the first two of these conditions cf. Bolza, *loc. cit.*, p. 538, ex. 29.

jectivity is in fact a perspectivity, the center C of the perspectivity being a point on OQ at the distance $\frac{|\lambda| \sin \omega}{\omega}$ from O .

Let R_1 and R_2 be the points on P_1Q and P_2Q determined by P_2C and P_1C , respectively. Then the condition

$$-\frac{\lambda}{r_1} - \frac{D_2}{D_1} < 0$$

means that r_1 is not on the segment P_1R_1 . Similarly, the condition

$$-\frac{\lambda}{r_2} - \frac{D_2}{D_1} < 0$$

being fulfilled means that r_2 does not lie on P_2R_2 .

Suppose now that a particular value r'_1 of r_1 be given, determining the point r'_1 in the figure. Draw r'_1C cutting P_2Q at S . Then the condition

$$\left(\frac{\lambda}{r_1} + \frac{D_2}{D_1}\right)\left(\frac{\lambda}{r_2} + \frac{D_2}{D_1}\right) - \frac{D_2^2}{D_1^2} > 0$$

means that r'_2 may not lie on the segment P_2S .

It remains to prove the statement made above, that (43) relates perspectively the points of P_1Q to those of P_2Q . In the figure take OX as the positive x -axis, O being the origin, of a set of perpendicular axes. Then P_1 is the point $(|\lambda| \cos \omega, -|\lambda| \sin \omega)$, while P_2 is $(|\lambda| \cos \omega, |\lambda| \sin \omega)$ and C is $\left(\frac{|\lambda| \sin \omega}{\omega}, 0\right)$. The point on P_1Q at the distance r_1 from P_1 is

$$(|\lambda| \cos \omega + r_1 \sin \omega, -|\lambda| \sin \omega + r_1 \cos \omega),$$

and the point on P_2Q at the distance r_2 from P_2 is

$$(|\lambda| \cos \omega + r_2 \sin \omega, |\lambda| \sin \omega - r_2 \cos \omega).$$

The condition that these two latter points be collinear with C is

$$\begin{vmatrix} |\lambda| \cos \omega + r_1 \sin \omega & |\lambda| \cos \omega + r_2 \sin \omega & \frac{|\lambda| \sin \omega}{\omega} \\ -|\lambda| \sin \omega - r_1 \cos \omega & |\lambda| \sin \omega - r_2 \cos \omega & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

After expansion and reduction this condition is found to be equivalent to relation (43).